SUPERSYMMETRIC W ALGEBRAS

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Based on collaboration work in progress with Prof. Y. Matsuo and Dr. K. Harada

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We start from the denition of the ane Yangian Y ([gl(1)) by Drinfeld currents

Affine Yangian

$$e(z) = \sum_{j=0}^{\infty} \frac{e_j}{z^{j+1}} , \qquad f(z) = \sum_{j=0}^{\infty} \frac{f_j}{z^{j+1}} , \qquad \psi(z) = 1 + \sigma_3 \sum_{j=0}^{\infty} \frac{\psi_j}{z^{j+1}} ,$$

Affine Yangian

$$e(z) = \sum_{j=0}^{\infty} \frac{e_j}{z^{j+1}} \,, \qquad f(z) = \sum_{j=0}^{\infty} \frac{f_j}{z^{j+1}} \,, \qquad \psi(z) = 1 + \sigma_3 \sum_{j=0}^{\infty} \frac{\psi_j}{z^{j+1}} \,,$$

$$\begin{aligned}
e(z)\, e(w) &\sim \varphi_3(z-w)\, e(w)\, e(z) \\
f(z)\, f(w) &\sim \varphi_3^{-1}(z-w)\, f(w)\, f(z) \\
\psi(z)\, e(w) &\sim \varphi_3(z-w)\, e(w)\, \psi(z) \\
\psi(z)\, f(w) &\sim \varphi_3^{-1}(z-w)\, f(w)\, \psi(z)
\end{aligned}$$

$$e(z)\, f(w) - f(w)\, e(z) = -\frac{1}{\sigma_3} \frac{\psi(z) - \psi(w)}{z - w} \,,$$

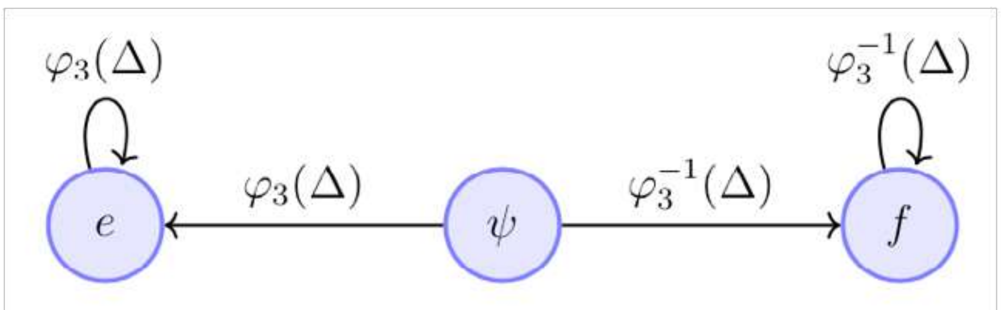
Affine Yangian

$$e(z)=\sum_{j=0}^{\infty}\frac{e_j}{z^{j+1}}\,,\qquad f(z)=\sum_{j=0}^{\infty}\frac{f_j}{z^{j+1}}\,,\qquad \psi(z)=1+\sigma_3\sum_{j=0}^{\infty}\frac{\psi_j}{z^{j+1}}\,,$$

$$\begin{aligned} e(z)\,e(w) &\sim \varphi_3(z-w)\,e(w)\,e(z)\\ f(z)\,f(w) &\sim \varphi_3^{-1}(z-w)\,f(w)\,f(z)\\ \psi(z)\,e(w) &\sim \varphi_3(z-w)\,e(w)\,\psi(z)\\ \psi(z)\,f(w) &\sim \varphi_3^{-1}(z-w)\,f(w)\,\psi(z)\end{aligned}$$

$$e(z)\,f(w)-f(w)\,e(z)=-\frac{1}{\sigma_3}\,\frac{\psi(z)-\psi(w)}{z-w}\,,$$

$$\varphi_3(z)=\frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)}=\frac{z^3+\sigma_2 z+\sigma_3}{z^3+\sigma_2 z-\sigma_3}$$



Affine Yangian

$$e(z) = \sum_{j=0}^{\infty} \frac{e_j}{z^{j+1}} , \qquad f(z) = \sum_{j=0}^{\infty} \frac{f_j}{z^{j+1}} , \qquad \psi(z) = 1 + \sigma_3 \sum_{j=0}^{\infty} \frac{\psi_j}{z^{j+1}} ,$$

$$J_1 = -f_0 , \qquad J_{-1} = e_0 ,$$

$$L_1 = -f_1 , \qquad L_{-1} = e_1$$

Virasoro algebra

energy-momentum tensor corresponding to the string action

$$T_{ab} = -4\pi\alpha' \frac{1}{\sqrt{g}} \frac{\delta L}{\delta g^{ab}}.$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n+1)(n-1)\delta_{n,-m}$$

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} T'(w) + \cdots$$

$$T(z)J(w) = \frac{1}{(z-w)^2} J(w) + \frac{1}{z-w} J'(w) + \cdots ;$$

W algebra

Miura transformation

$$: \prod_{j=1}^{n} (Q\partial_z - h_j \cdot \partial\varphi) := \sum_{d=0}^{n} W^{(d)}(z)(Q\partial_z)^{n+1-d}.$$

$$h_i = e_i - \frac{1}{n}\sum_{i=1}^{n} e_i$$

W algebra

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\
[L_n, W_m] &= (2n-m)W_{n+m} \\
\frac{2}{9}[W_n, W_m] &= \frac{c}{3\cdot 5!}n(n^2-1)(n^2-4)\delta_{n+m,0} + \frac{16}{22+5c}(n-m)\Lambda_{n+m} \\
&\quad +(n-m)\left(\frac{1}{15}(n+m+2)(n+m+3) - \frac{1}{6}(n+2)(m+2)\right)L_{n+m}
\end{aligned}$$

$$\Lambda_n = \sum_{k=-\infty}^{\infty} :L_k L_{n-k}: + \frac{1}{5}x_n L_n\,; \quad x_{2l} = (1+l)(1-l)\,, \quad x_{2l+1} = (2+l)(1-l)$$

Supersymmetric Virasoro algebra

$$
\begin{aligned}
T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}T'(w) + \cdots\,, \\
T(z)G(w) &= \frac{3/2}{(z-w)^2}G(w) + \frac{1}{z-w}G'(w) + \cdots\,, \\
G(z)G(w) &= \frac{2c/3}{(z-w)^3} + \frac{2}{z-w}T(w) + \cdots\,,
\end{aligned}
$$

Supersymmetric *N*=2 Virasoro algebra

$$\begin{aligned}
T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}T'(w) + \cdots\,, \\
T(z)G^{\pm}(w) &= \frac{3/2}{(z-w)^2}G^{\pm}(w) + \frac{1}{z-w}G^{\pm\prime}(w) + \cdots\,, \\
T(z)J(w) &= \frac{1}{(z-w)^2}J(w) + \frac{1}{z-w}J'(w) + \cdots\,, \\
J(z)G^{\pm}(w) &= \frac{\pm}{z-w}G^{\pm}(w) + \cdots\,, \\
G^{+}(z)G^{-}(w) &= \frac{c/3}{(z-w)^3} + \frac{1}{(z-w)^2}J(w) + \frac{1}{z-w}T(w) + \frac{1/2}{z-w}J'(w) + \cdots\,, \\
J(z)J(w) &= \frac{c/3}{(z-w)^2} + \cdots\,,
\end{aligned}$$

Supersymmetric *N*=2 W algebra

$$\varphi(z) = \left(\varphi^{(1)}(z), \varphi^{(2)}(z), \dots, \varphi^{(2n+1)}(z)\right)$$
$$D = (\partial/\partial\theta) + \theta(\partial/\partial z)$$
$$\Phi(Z) = \varphi(z) + \sqrt{-1}\theta\chi(z)$$

$$\Theta_i(Z) = (-1)^{i-1} h_i \cdot D\Phi(Z)$$

Lax operator $$L(Z) =: \prod_{j=1}^{2n+1} (aD - \Theta_j(Z)) :$$

$$L(Z) = \sum_{d=0}^{2n+1} W^{(d/2)}(Z)(aD)^{2n+1-d}$$

Supersymmetric *N*=2 W algebra

Lax operator $L(Z) =: \prod_{j=1}^{2n+1} (aD - \Theta_j(Z)) :$

$$L(Z) = \sum_{d=0}^{2n+1} W^{(d/2)}(Z)(aD)^{2n+1-d}$$

$$W_1(Z) \quad = \quad J(z) + i\theta\big(G^+(z) - G^-(z)\big)$$

$$\frac{1}{a}W_{3/2}(Z) \quad = \quad iG^-(z) + \theta\big(T(z) + \frac{1}{2}\partial J(z)\big)$$

Supersymmetric *N*=2 Virasoro algebra

T

G^{+} G^{-}

J

Supersymmetric *N*=2 W algebra

W^{21}

W^{2+} W^{2-}

T W^{20}

G^{+} G^{-}

J

Supersymmetric *N*=2 W algebra

W^{31}

W^{3+} W^{3-}

W^{30}

W^{21}

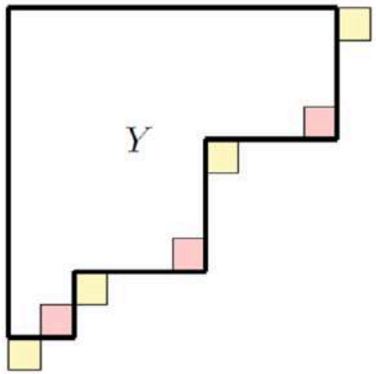
W^{2+} W^{2-}

T W^{20}

G^{+} G^{-}

J

plane partition representation for affine Yangian



$\square \in A(Y)$ (yellow)

$\square \in R(Y)$ (pink)

$$\begin{aligned} \psi(z)|\Lambda\rangle &= \psi_\Lambda(z)|\Lambda\rangle \,, \\ e(z)|\Lambda\rangle &= \sum_{\square\in \mathrm{Add}(\Lambda)} \frac{T_{\square}(\Lambda)}{z-h(\square)}|\Lambda+\square\rangle \,, \\ f(z)|\Lambda\rangle &= \sum_{\square\in \mathrm{Rem}(\Lambda)} \frac{T_{\square}(\Lambda)}{z-h(\square)}|\Lambda-\square\rangle \,, \end{aligned}$$

$\psi(z)$ acts diagonally on a plane partition configuration Λ with eigenvalue

$$\psi_\Lambda(z) = \left(1+\frac{\psi_0\sigma_3}{z}\right) \prod_{\square\in\Lambda} \varphi_3(z-h(\square)) \,,$$

where

$$h(\square) = h_1 x_1(\square) + h_2 x_2(\square) + h_3 x_3(\square)$$

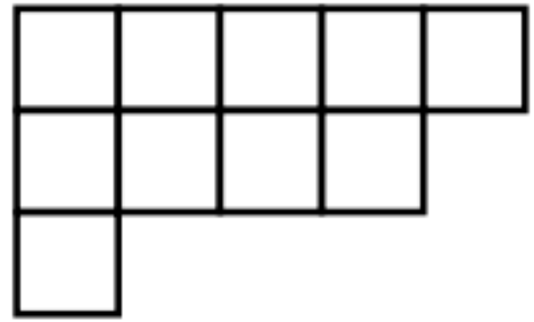
with $x_1(\square)$ the x_1-coordinate of the box, etc.

$$T_{\square}(\Lambda) = \left[-\frac{1}{\sigma_3}\mathrm{Res}_{w=h(\square)}\psi_\Lambda(w)\right]^{\frac{1}{2}}$$

partition

5
4 + 1
3 + 2
3 + 1 + 1
2 + 2 + 1
2 + 1 + 1 + 1
1 + 1 + 1 + 1 + 1

Young diagram



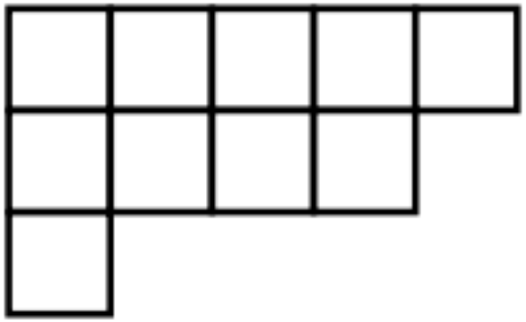
The generating function for p(n) is given by

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \left(\frac{1}{1 - x^k} \right)$$

partition

5
4 + 1
3 + 2
3 + 1 + 1
2 + 2 + 1
2 + 1 + 1 + 1
1 + 1 + 1 + 1 + 1

Young diagram

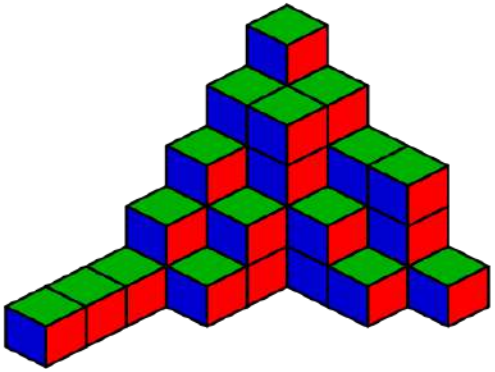


The generating function for p(n) is given by

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty}\left(\frac{1}{1-x^k}\right)$$

plane partition

$$1 \quad 1 \quad 1 \qquad \begin{matrix} 1 & 1 \\ 1 & \end{matrix} \qquad \begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \qquad 2 \quad 1 \qquad \begin{matrix} 2 \\ 1 \end{matrix} \qquad 3$$



the generating function for PL(n) is given by

$$\sum_{n=0}^{\infty} \mathrm{PL}(n)\, x^n = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)^k} = 1 + x + 3x^2 + 6x^3 + 13x^4 + 24x^5 + \cdots$$

Plane partition representation for N=2 affine Yangian

$$\begin{aligned} P(z)|\Lambda\rangle &= P_\Lambda(z)|\Lambda\rangle \ , \\ x(z)|\Lambda\rangle &= \sum_{\blacksquare \in \mathrm{AdI}(\Lambda)} \frac{S_\blacksquare(\Lambda)}{z-p(\blacksquare)}|\Lambda+\blacksquare\rangle \ , \\ y(z)|\Lambda\rangle &= \sum_{\blacksquare \in \mathrm{ReI}(\Lambda)} \frac{S_\blacksquare(\Lambda)}{z-p(\blacksquare)}|\Lambda-\blacksquare\rangle \ , \end{aligned}$$

The eigen value

$$P_\Lambda(z) = K\left\{\prod_{\blacksquare\in\lambda} P_\blacksquare(u) \prod_{\square\in\mathcal{E}} P_\square(u)\right\} \ ,$$

with

$$P_\square(z) = \tau_2(z-h(\square)) \ .$$

and

$$P_\blacksquare(z) = \prod_{k=0}^{m+n-1} \tau_2\big(z-g(\blacksquare)-kh_2\big)$$

Here

$$\tau_2(u) = \frac{(u+h_1)(u+h_3)}{u(u-h_2)} \ .$$

we see $\varphi_2(u)\tau_2(-u) = 1$.

$$S_\blacksquare(\Lambda) = \left[\mathrm{Res}_{w=h(\blacksquare)} P_\Lambda(w)\right]^{\frac{1}{2}}$$

$$g(\blacksquare) \quad \equiv \quad x_1(\blacksquare)h_1 + x_3(\blacksquare)h_3 \ ,$$

and

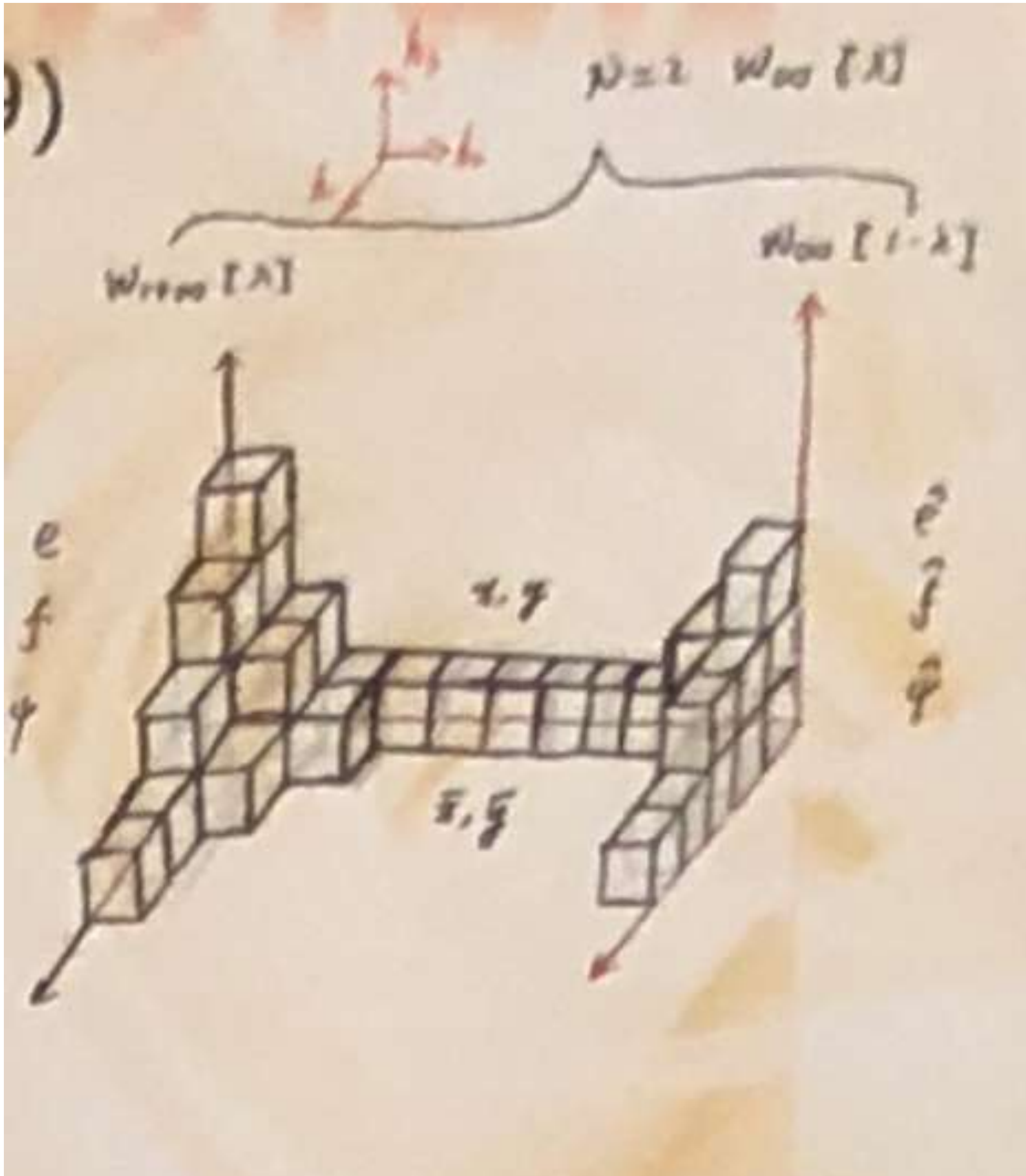
$$h(\blacksquare) \equiv x_1(\blacksquare)h_1 + x_3(\blacksquare)h_3 + \big(x_1(\blacksquare) + x_3(\blacksquare)\big)h_2 = -x_3(\blacksquare)h_1 - x_1(\blacksquare)h_3$$

Then the pole postion is

$$p(\blacksquare) \equiv h(\blacksquare) + \ell h_2 \ ,$$

where ℓ is the number of additional boxes extending the minimal length bud. Here the minimal length bud arise from that, in order for x to be allowed to add an infinite row along the x_2 direction at $(x_1, x_3) = (m, n)$, there must be at least a bud of $m + n$ boxes extending in the x_2 direction at that position, i.e. the box configuration must already contain boxes at

$$(x_1, x_2, x_3) = (m, 0, n), (m, 1, n), (m, 2, n), \ldots, (m, m + n - 1, n) \ .$$



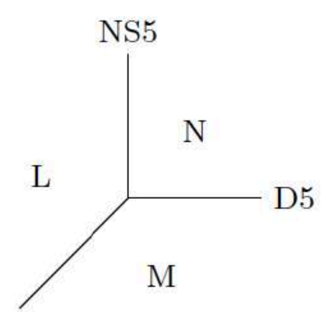
Relation with Vertex operator algebra

$$\sum_{i=1}^{3} \lambda_i^{-1} = 0.$$

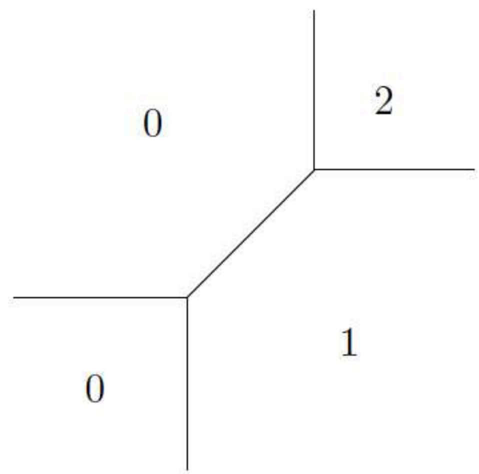
When λ_i satisfies the extra condition

$$\frac{L}{\lambda_1} + \frac{M}{\lambda_2} + \frac{N}{\lambda_3} = 1,$$

the basis $|\Lambda\rangle$ which contains a box with a coordinate $(L+1, M+1, N+1)$ becomes null. The corresponding Yangian is equivalent to the vertex operator algebra $Y_{L,M,N}[\Psi]$.



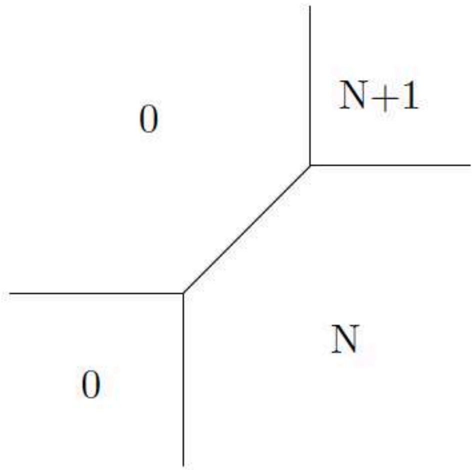
For example, the following diagram represents $\mathcal{N}=2$ super Virasoro algebra $\otimes$ $U(1)$ current:



The parameters of two Yangians are,

$$\lambda_1^{(1)} = -\frac{n}{n+2}, \quad \lambda_2^{(1)} = \frac{n}{n+1}, \quad \lambda_3^{(1)} = n,$$
$$\lambda_1^{(2)} = -\frac{1}{n+2}, \quad \lambda_2^{(2)} = \frac{1}{n+1}, \quad \lambda_3^{(2)} = 1.$$

And for $\mathcal{N}=2$ W algebra:



As solutions of (10) and (11), the parameters of two Yangians are,

$$\lambda_1^{(1)} = -\frac{k}{k+N+1}, \quad \lambda_2^{(1)} = \frac{k}{k+N}, \quad \lambda_3^{(1)} = k,$$

$$\lambda_1^{(2)} = -\frac{N}{k+N+1}, \quad \lambda_2^{(2)} = \frac{N}{k+N}, \quad \lambda_3^{(2)} = N.$$

For $Y_{N,N+1,0}$ (resp. $Y_{0,0,N}$). (Actually there are many other choices, i.e.,$\lambda_1^{(2)} = -\frac{N}{n+2}, \quad \lambda_2^{(2)} = \frac{N}{n+1}$, But the above choice we use shows a symmetry between N and k.)

Plane partition realization of free WoW

Bosonic ghost

We start from the bosonic ghost fields $\beta(z) = \sum_{n\in\mathbb{Z}} \beta_n z^{-n}$, $\gamma(z) = \sum_{n\in\mathbb{Z}} \gamma_n z^{-n-1}$. The commutation among the oscillators is given by $[\beta_n, \gamma_m] = \delta_{n+m,0}$.

$$\beta_{-n_1} \cdots \beta_{-n_g} \gamma_{-m_1} \cdots \gamma_{-m_g} |0\rangle$$

$$\beta_{-n_1} \cdots \beta_{-n_g} \gamma_{-m_1} \cdots \gamma_{-m_g} |0\rangle \quad =$$

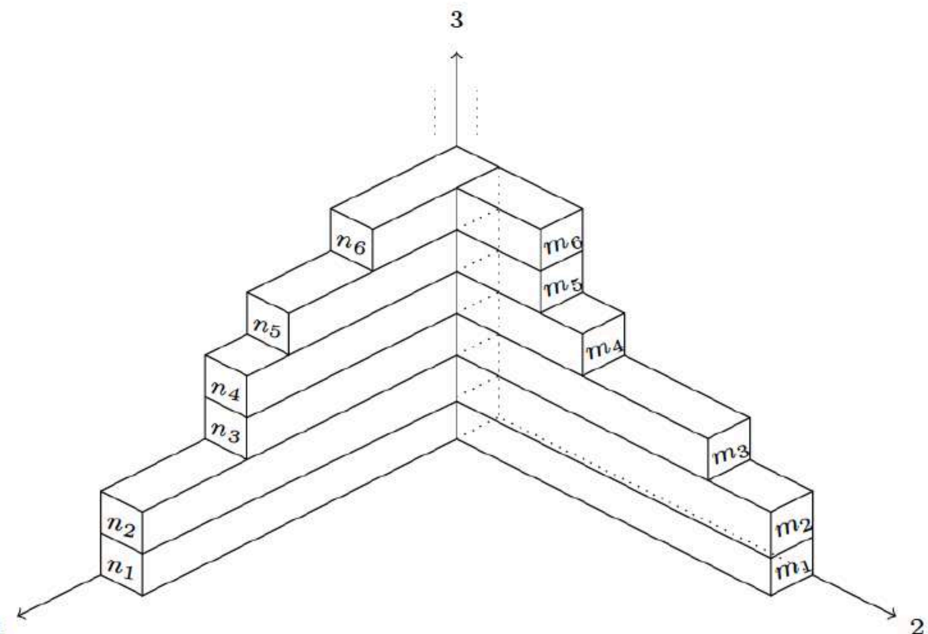


Figure 1: The relation between Hilbert space of $\beta\gamma$ ghost and plane partition with a pit at (2,2,1). It can be decomposed into two Young diagrams. The left (right) one corresponds to the Hilbert space of β (γ) ghost. The number written in each row means its length.

The partition function becomes,

$$\chi(q) = \sum_{g=0}^{\infty} \sum_{n_1 \geq \cdots \geq n_g \geq 1} \sum_{m_1 \geq \cdots \geq m_g \geq 0} q^{\sum_i (n_i + m_i)} = \sum_{n=0}^{\infty} \frac{q^n}{((q;q)_n)^2}, \qquad (a;q)_n = \prod_{m=0}^{n-1} (1 - aq^m) .$$

We have this as a result of

$$\sum_{m_1 \geq \cdots \geq m_g \geq 0} q^{\sum_i (m_i)} = \prod_{k=1}^{g} \frac{1}{(1-q^k)} = \frac{1}{(q;q)_g},$$

and by setting $n_i = n'_i + 1$,

$$\sum_{n_1 \geq \cdots \geq n_g \geq 1} q^{\sum_i (n_i)} = \sum_{n'_1 \geq \cdots \geq n'_g \geq 0} q^{g + \sum_i (n'_i)} .$$

Free fermion

$\{b_n, c_m\} = \delta_{n+m,0}$ is spanned by

$$b_{-n_1} \cdots b_{-n_g} c_{-m_1} \cdots c_{-m_g} |0\rangle$$

The partition function becomes

$$\chi(q) = \sum_{g=0}^{\infty} \sum_{n_1 > \cdots > n_g \geq 1} \sum_{m_1 > \cdots > m_g \geq 0} q^{\sum_i (n_i + m_i)} = \sum_{n=0}^{\infty} \frac{q^{n^2/2}}{((q;q)_n)^2} = \prod_{n=1}^{\infty} (1 - q^n)^{-1} .$$

Where the following is applied. By setting $n_i = n'_i + g + 1 - i$,

$$\sum_{n_1 > \cdots > n_g \geq 1} q^{\sum_i (n_i)} = \sum_{n'_1 \geq \cdots \geq n'_g \geq 0} q^{\frac{1}{2} g(g+1) + \sum_i (n'_i)} ,$$

and by setting $m_i = m'_i + g - i$,

$$\sum_{m_1 > \cdots > m_g \geq 0} q^{\sum_i (m_i)} = \sum_{m'_1 \geq \cdots \geq m'_g \geq 0} q^{\frac{1}{2} g(g-1) + \sum_i (m'_i)} .$$

$$b_{-n_1}\cdots b_{-n_g}c_{-m_1}\cdots c_{-m_g}\,|0\rangle \quad =$$

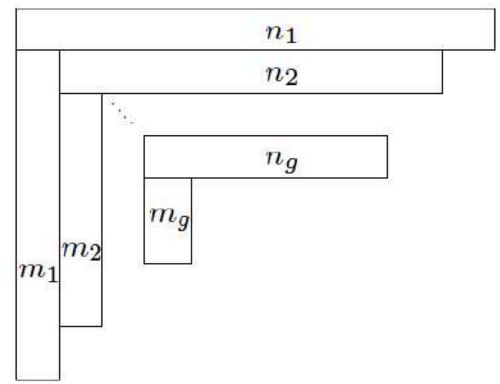


Figure 2: The relation between Hilbert space of *bc* ghost and Young diagram (plane partition with a pit at (1,1,2)). The number written in each row or column means its length.

WoW: N = 2 W algebra

$$\beta_{-n_1}\cdots\beta_{-n_{g1}}\gamma_{-m_1}\cdots\gamma_{-m_{g2}}b_{-\bar{n}_1}\cdots b_{-\bar{n}_{g3}}c_{-\bar{m}_1}\cdots c_{-\bar{m}_{g4}}|0\rangle$$

with $g_1+g_3=g_2+g_4$ but $g_1\neq g_2$, $g_3\neq g_4$ in general.

The partition function with the summation over the infinite leg becomes

$$\chi[q]=\sum_{g_1,g_2,g_3,g_4\geq 0, g_1+g_3=g_2+g_4}\ \sum_{n_1\geq\cdots\geq n_{g_1}\geq 1}\ \sum_{m_1\geq\cdots\geq m_{g_2}\geq 0}\ \sum_{\bar{n}_1>\cdots>\bar{n}_{g_3}\geq 1}\ \sum_{\bar{m}_1>\cdots>\bar{m}_{g_4}\geq 0} q^{\sum_i^{g_1} n_i+\sum_i^{g_2} m_i+\sum_i^{g_3}\bar{n}_i+\sum_i^{g_4}\bar{m}_i}$$

$$=\sum_{g_1,g_2,g_3,g_4\geq 0, g_1+g_3=g_2+g_4} q^{g_1+\frac{1}{2}g_3(g_3+1)+\frac{1}{2}g_4(g_4-1)}\prod_{i=1}^{4}(q;q)_{g_i}^{-1}=\prod_{n=1}^{\infty}\frac{(1+q^n)^2}{(1-q^n)^2}$$

It is straightforward to generalize the system to an arbitrary number of quartets $(b^{(I)},c^{(I)},\beta^{(I)},\gamma^{(I)})$ $(I=1,\cdots,M)$. Such system describes the $\mathcal{N}=2$ super W-algebra. In this case we have

$$g_1^{(I)}+g_3^{(I)}=g_2^{(I)}+g_4^{(I)}$$

$$\chi[q]=\sum_{g_1^{(I)},g_2^{(I)},g_3^{(I)},g_4^{(I)}\geq 0, g_1^{(I)}+g_3^{(I)}=g_2^{(I)}+g_4^{(I)}} q^{\sum_{I=1}^{M}\left(g_1^{(I)}+\frac{1}{2}g_3^{(I)}(g_3^{(I)}+1)+\frac{1}{2}g_4^{(I)}(g_4^{(I)}-1)\right)}\prod_{I=1}^{M}\prod_{i=1}^{4}(q;q)_{g_i^{(I)}}^{-1}$$

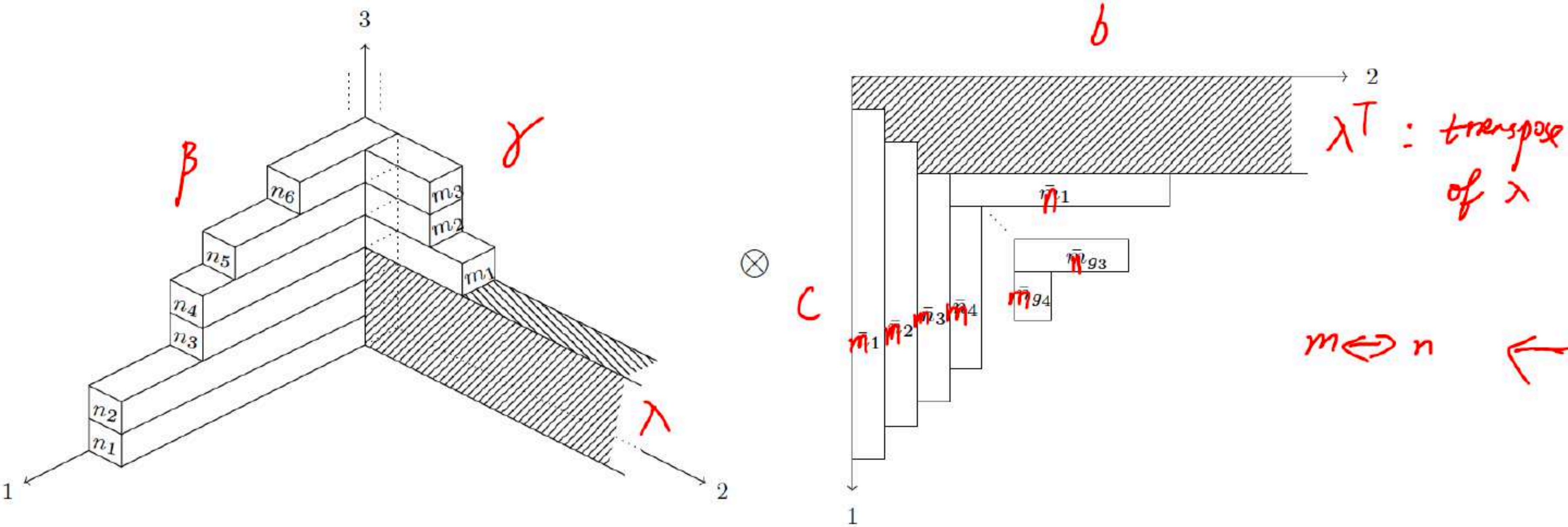


Figure 3: The figure represents the state in the form of (3.5) with $g_1 - g_2 = g_4 - g_3 > 0$. The rows with infinite length and height $g_1 - g_2$ are inserted. The above case corresponds to $g_1 - g_2 = 3$.

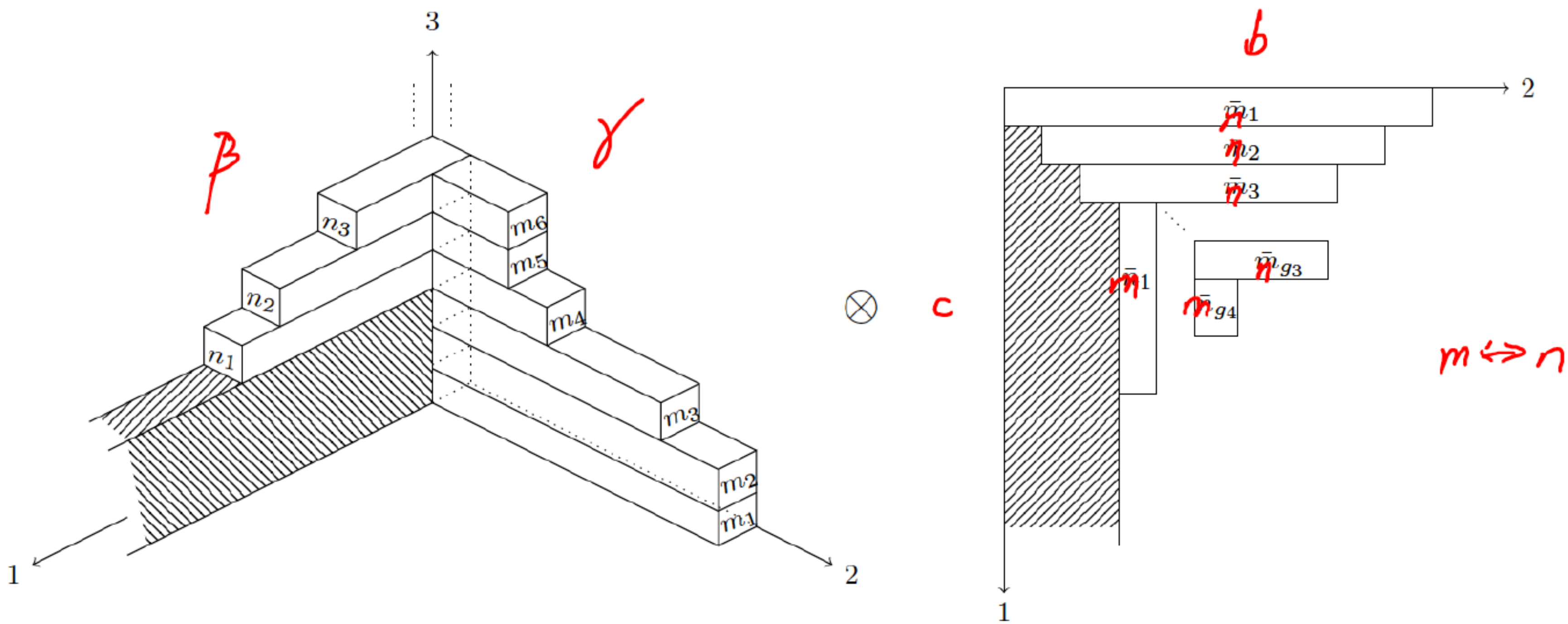


Figure 4: The figure represents the state in the form of (3.5) with $g_1 - g_2 = g_4 - g_3 < 0$. The rows with infinite length and height $g_2 - g_1$ are inserted. The above case corresponds to $g_2 - g_1 = 3$.

Action of free field operators

$$\psi_r = \sum_{m\in\mathbb{Z}+\frac{1}{2}} \sum_i \Big((-m-\tfrac{1}{2})^r - (-m+\tfrac{1}{2})^r \Big) : c^i_{-m+\frac{1}{2}} b^i_{m-\frac{1}{2}} :$$

$$e_r = -\sum_{m\in\mathbb{Z}+\frac{1}{2}} \sum_i (-m-\tfrac{1}{2})^r : c^i_{-m-\frac{1}{2}} b^i_{m-\frac{1}{2}} :$$

$$f_r = \sum_{m\in\mathbb{Z}+\frac{1}{2}} \sum_i (-m+\tfrac{1}{2})^r : c^i_{-m+\frac{3}{2}} b^i_{m-\frac{1}{2}} : ,$$

$$\hat{\psi}_r = \sum_{m\in\mathbb{Z}} \sum_i \Big(-(m+1)(-m)^{r-1} + m\big(-m+1\big)^{r-1} \Big) : \beta^i_{-m} \gamma^i_m :$$

$$\hat{e}_r = -\sum_{m\in\mathbb{Z}} \sum_i (-m)^r : \beta^i_{-1-m} \gamma^i_m :$$

$$\hat{f}_r = -\sum_{m\in\mathbb{Z}} \sum_i m\big(-m+1\big)^{r-1} : \beta^i_{1-m} \gamma^i_m : \; .$$

$$x_s = \sum_{m\in\mathbb{Z}} \sum_i m(-m-1)^{s-\frac{1}{2}} \ :c^i_{-1-m}\,\beta^i_m: ,$$

and

$$\bar{x}_s = \sum_{m\in\mathbb{Z}} \sum_i (m+1)^{s-\frac{1}{2}} \ :b^i_{-2-m}\,\gamma^i_m: ,$$

where $s = \frac{1}{2}, \frac{3}{2}, \ldots$

With those defination it is obvious to see that, e_r adds one box to the free fermion basis, f_r reomoves it, and ψ_r is an eigen operator. Likewise, $\hat{e}_r$, $\hat{f}_r$ and $\hat{\psi}_r$ do the same on the bosonic ghost basis.

Also we know

$$G^+_n = \sqrt{2}\sum_{i=1}^{N}\{\sum_{m\in\mathbb{Z}} m : c^i_{n-m+\frac{1}{2}}\beta^i_m : + a\,(-n-\frac{1}{2})c^i_{n+\frac{1}{2}}\}$$

So $x_{\frac{1}{2}} = \frac{1}{\sqrt{2}}G^+_{-\frac{3}{2}}$ at the free field limit $a = 0$.

The OPE of e and x

$$
\begin{aligned}
\psi_r &= \sum_{m\in\mathbb{Z}}\sum_i \Big((-m-1)^r-(-m)^r\Big)\ :c^i_{-m}b^i_m: \\
e_r &= -\sum_{m\in\mathbb{Z}}\sum_i (-m-1)^r\ :c^i_{-m-1}b^i_m: \\
f_r &= \sum_{m\in\mathbb{Z}}\sum_i (-m)^r\ :c^i_{-m+1}b^i_m:\ ,
\end{aligned}
$$

If we choose $x_{\frac{1}{2}}=\frac{1}{\sqrt{2}}G^+_{-\frac{1}{2}}$, then

$$x_s=-\sum_{m\in\mathbb{Z}}\sum_i(-m)^{s+\frac{1}{2}}\ :c^i_{-m}\,\beta^i_m:\ ,$$

with these definition we can check that at

$$[e_{j+2},e_k]-2[e_{j+1},e_{k+1}]+[e_{j+1},e_{k+2}]-[e_{j+1},e_k]+[e_j,e_{k+1}]=0$$

$$[e_{j+1},x_k]-[e_j,x_{k+1}]+[e_j,x_k]=0$$

$$|\Lambda\rangle = \beta^i_{-n_1}\cdots\beta^i_{-n_{g_1}}\gamma^i_{-m_1}\cdots\gamma^i_{-m_{g_2}}b^i_{-\bar{n}_1}\cdots b^i_{-\bar{n}_{g_3}}c^i_{-\bar{m}_1}\cdots c^i_{-\bar{m}_{g_4}}|0\rangle \equiv |\vec{Y}_1;\vec{Y}_2;\vec{Y}_3;\vec{Y}_4:\lambda\rangle \equiv |\Lambda_L;\Lambda_R:\lambda\rangle$$

where λ is the asymptotic Young diagram along the x_2 axis, and in the case of $i=1$, λ is a single column with height $g_1-g_2=h$. Λ_L stands for the left plane partition constructed by $\vec{Y}_1;\vec{Y}_2$. Λ_R stands for the right one.

$$x_s|\Lambda\rangle = -\sum_{m\geq 0}\sum_i(-m)^{s+\frac{1}{2}}|\vec{Y}_1;\vec{Y}_2-m;\vec{Y}_3;\vec{Y}_4+m:\lambda+\square\rangle - \sum_{m<0}\sum_i(-m)^{s+\frac{1}{2}}|\vec{Y}_1+|m|;\vec{Y}_2;\vec{Y}_3-|m|;\vec{Y}_4:\lambda+\square\rangle$$

where $\vec{Y}_2-m$ stands for removing γ_{-m} from $\vec{Y}_2$, and $\vec{Y}_1+|m|$ means adding $\beta_{-|m|}$ to $\vec{Y}_1$.

$$e_r|\Lambda\rangle = -\sum_{m\geq 0}\sum_i(-m-1)^r\,|\vec{Y}_1;\vec{Y}_2;\vec{Y}_3;\vec{Y}_4-m+(m+1):\lambda\rangle - \sum_{m<0}\sum_i(-m-1)^r|\vec{Y}_1;\vec{Y}_2;\vec{Y}_3-|m-1|+|m|;\vec{Y}_4:\lambda\rangle$$

According to the discussion of the coordinate systems, we see the factor $(-m-1)$ exactly corresponds to the two kinds of poles adding to $\bar{m}_i$ and $\bar{n}_i$, thus the above can be rewritten as

$$e_r|\Lambda\rangle \;=\; -\sum_{\square\in\mathrm{Add}(\Lambda_R)}(h(\square))^r|\Lambda_L;\Lambda_R+\square\rangle\ ,$$

(we may check the minus sign in the front later)

$$e(z)|\Lambda\rangle \;=\; -\sum_{\square\in\mathrm{Add}(\Lambda_R)}\frac{1}{z-h(\square)}|\Lambda_L;\Lambda_R+\square\rangle\ ,$$

In this sense, we can recover their action away from the free field limit as (74), thus reproduce the definating OPE of the affine Yangian.